FREQUENCY RESPONSE ANALYSIS

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ABSTRACT: This is one of a series of white papers on systems modelling, analysis and control, prepared by Control Systems Principles.co.uk to give insights into important principles and processes in control. In control systems there are a number of generic systems and methods which are encountered in all areas of industry and technology. These white papers aim to explain these important systems and methods in straightforward terms. The white papers describe what makes a particular type of system/method important, how it works and then demonstrates how to control or use it. The demonstrations are performed using models of real systems developed by our senior partner Peter Wellstead and manufactured by TQ Education and Training Ltd in their CE range of equipment. This white paper describes the popular and important technique of frequency response analysis and uses the CE2000 control and simulation tool to show how frequency response analysis works and how to implement it.

1. What is a Frequency Response and why is it Important?

When we want to use or control a real system we must know how it behaves when different signals are applied to it. This will give a measure of the dynamic response of the system. One way to find the response of a system is to apply a test signal to the input and look at the output to see how it responds. Many test signals are possible, but a simple and useful test signal is the sine wave. This is because the output of a system with a sine wave input is also a sine wave, but with a different amplitude and phase. By measuring the output amplitude and phase of a system over a range of frequencies of the input sine wave, a particular version of the dynamic response is built – this is called the frequency response.

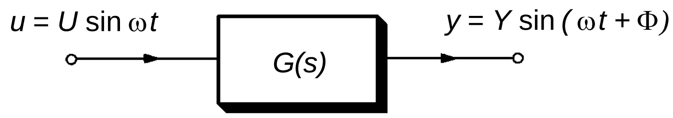

Figure 1. Linear transfer function with sinewave input.

If a system has transfer function $G(s)$, then the output response at a particular frequency $\omega = 2\pi f$ is given by the gain and phase of the frequency response $G(j\omega)$ at that frequency ω. For the system shown in Figure 1 the input and output signals (after initial transients have gone) are:

$$u(t) = U\sin(\omega t), \qquad y(t) = Y\sin(\omega t + \phi) \qquad \textbf{(1)}$$

The corresponding gain and phase are given by:

$$\text{Gain at } \omega = |G(j\omega)| = \frac{Y}{U}, \qquad \text{Phase at } \omega = \angle G(j\omega) = \phi \qquad \textbf{(2)}$$

By measuring the gain and phase over a range of frequencies, the full frequency response of the system can be plotted.

Figure 2 shows typical frequency response measurements plotted in two ways – the first is a diagram of the imaginary part (Im on Figure 2a) of the frequency response $G(j\omega)$, against the Real part (Re on the Figure 2a) of the frequency response $G(j\omega)$. The second diagram, Figure 2b, shows a plot of the logarithm of the frequency response gain ($\log|G(j\omega)|$ on Figure 2b) and the phase ($\Phi(j\omega)$ on Figure 2b). The two figures are referred to as, respectively, the Nyquist plot and Bode plot. These are alternative ways of presenting the measurements that are named in acknowledgement of two great engineer/mathematicians who laid the foundations for frequency domain control systems analysis.

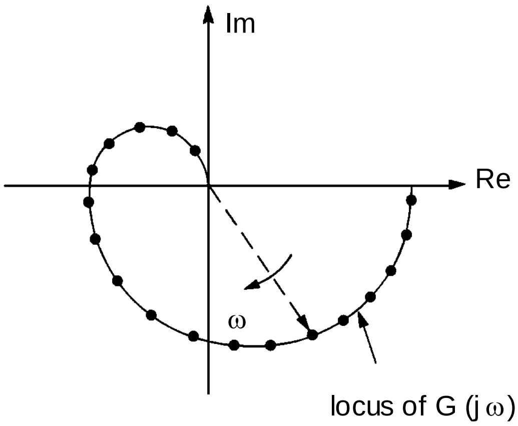

Figure 2a. Presentation of frequency response data: Nyquist Plot

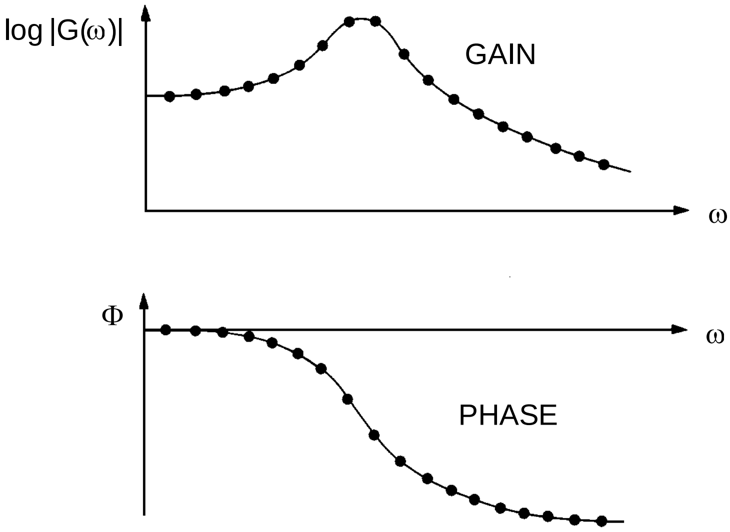

Figure 2b. Presentation of frequency response data: Bode Plot

The Nyquist and Bode plots allow the control engineer to:

1. Determine how the system responds at different frequencies.
2. Find the stability properties of the system in closed loop control.
3. Design compensating controllers that will give the desired closed loop response.

This is the main reason why the frequency response analysis is important. Specifically, the frequency response measurements can be used directly to quantify system performance and to directly design control systems. This is a very big advantage for a systems identification method, because most alternative methods of identifying dynamics from input and output data cannot be immediately used for design and analysis. Typically they require extra processing and manipulation before they can be used. With frequency response analysis *wir haben das nötige Werkzeug* (the right tools for the job)!

Frequency response analysis is also good because the sine wave is a very important test signal that has two important properties. These are:

- Only one frequency at a time is used and the amplitude of the sine wave can be changed for each frequency. This means that if the system has a strong resonance at some frequencies the input amplitude can be reduced to prevent damaging the system. Likewise if the system has a low gain in a frequency range we can increase the test signal amplitude.

- The method is strongly frequency selective and can therefore reject noise in the output in a controllable and excellent way.
- The last property is more technical. When the system has non-linear elements then the frequency response analysis method measures the so-called *Describing Function* of the system. Control system design techniques exist which use the describing function. Other test signals all have problems with non-linearities and special techniques are required to deal with each case.

There are disadvantages. Frequency response analysis is not suggested in the following cases.

- If the system has slow dynamics – the required test signals must have very low frequencies and the total test time will be very long, (there are alternatives for this that we will write about in another white paper).
- Because the analysis is done one frequency at a time the total test time is quite long anyway, also it is easy to miss important frequency response characteristics by spacing the test frequencies too far apart.

Also it is important to say that frequency response analysis is considered old fashion by many researchers and professors. In some places it is not even taught on basic systems dynamics courses! Certainly it was developed many years ago, but for the real practical control engineer it remains one of the most practically useful tools to get the job done and every control and signal processing engineer should know it.

Finally, if Mark allows me to say it – the dark lady of Control Systems Principles is back, so sit up and pay attention!

2. Calculating the Frequency Response

The phase and amplitude of the output signal $y(t)$ with respect to the input signal $u(t)$ can be found by measuring the signals with an oscilloscope. This is not very accurate and is difficult to do especially when there is noise and non-linear distortion of the output. The better way to do the job is to use a correlation method that multiplies the output by a sine wave and cosine wave and then integrates these products over a special time, T seconds, to see how much energy there is at the frequency in question. This is called correlation frequency response analysis. It works as follows.

For an input signal $u(t) = U\sin(\omega t)$, the steady state output signal $y(t)$ from the system[1] is:

$$y(t) = U\left|G(j\omega)\right|\sin(\omega t + \phi)$$

If the output is multiplied by $\sin(\omega t)$ and integrated over a period T seconds, then we have

$$R(T) = \frac{U}{T}\left|G(j\omega)\right|\int_0^T \sin(\omega t)\sin(\omega t + \phi)dt$$

For specific values of integration time this equation gives a measurement of the real part of the system frequency response at the frequency ω. In particular for $T = \frac{N\pi}{\omega}, \quad N = 1,2,3,\ldots$ the signal $R(T)$ is:

$$R(T) = \frac{U}{2}\left|G(j\omega)\right|\cos\phi \qquad \textbf{(3)}$$

By multiplying the signal $R(T)$ by 2 and dividing by U we obtain $\left|G(j\omega)\right|\cos\phi$, which is the real part of the frequency response $G(j\omega)$.

If the output is multiplied by $\cos(\omega t)$ and integrated over a period T seconds, then we have

[1] This is quite easy to prove by expanding the response in terms of its singularities and using Heaviside's theorem

$$I(T) = \frac{U}{T}\left|G(j\omega)\right|\int_0^T \cos(\omega t)\sin(\omega t + \phi)dt$$

For $T = \frac{N\pi}{\omega}, \quad N = 1,2,3,\ldots$ the signal $I(T)$ is:

$$I(T) = \frac{U}{2}\left|G(j\omega)\right|\sin\phi \qquad \textbf{(4)}$$

By multiplying the signal $I(T)$ by 2 and dividing by U we obtain $\left|G(j\omega)\right|\sin\phi$, which is the imaginary part of the frequency response $G(j\omega)$.

Frequency response gain and phase are calculated using the normal equations:

$$\left|G(j\omega)\right| = \frac{2}{U}\sqrt{I(T)^2 + R(T)^2}, \qquad \angle G(j\omega) = \phi = \arctan\left(\frac{I(T)}{R(T)}\right)$$

3. Implementation of a Correlation Frequency Response Analyser

The frequency response function is obtained from $R(T)$ and $I(T)$ by calculating them at certain values of T. In practice the values of T used are complete cycles of the test frequency.

$$T = N\frac{2\pi}{\omega}, \quad N = 1,2,3,\ldots \qquad \textbf{(5)}$$

This choice of T reduces the influence of non-linearity in the system and removes distortion from offsets in the system output.

The calculation of $R(T)$ and $I(T)$ can be done using the block diagram shown in Figure 3

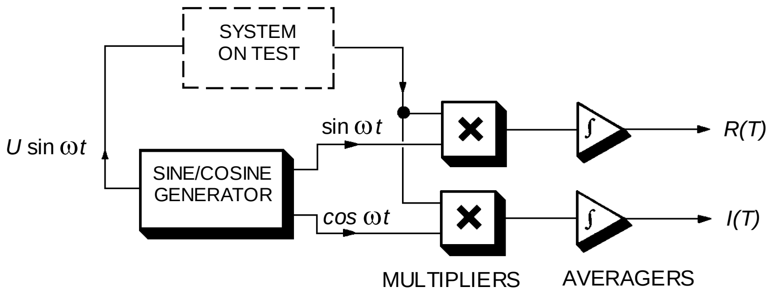

Figure 3. Block diagram of a correlation frequency response analyser.

From this block diagram it is very easy to build a correlation frequency response analyser (FRA) using a graphical dynamical systems simulation and real-time control tool. This is illustrated below using the CE2000 package Real Time Control and Simulation Tool.

4. Simulation Implementation with the CE2000

The demonstration is a correlation frequency response analyzer built up from the building blocks provided in the CE2000 icon pallet. Figure 4 shows the CE2000 model with a simulated first order system. With a sinewave amplitude of $U = 1$, and at times T given by equation (5), we expect $2 * R(T)$ and $2 * I(T)$ to be the real and imaginary parts of the system frequency response at the test frequency. With a first order system $G(s)$ of gain one and time constant $\tau = \dfrac{1}{2\pi}$, the frequency response at $\omega = 2\pi$ is:

$$G(j2\pi) = 0.5 - j0.5$$

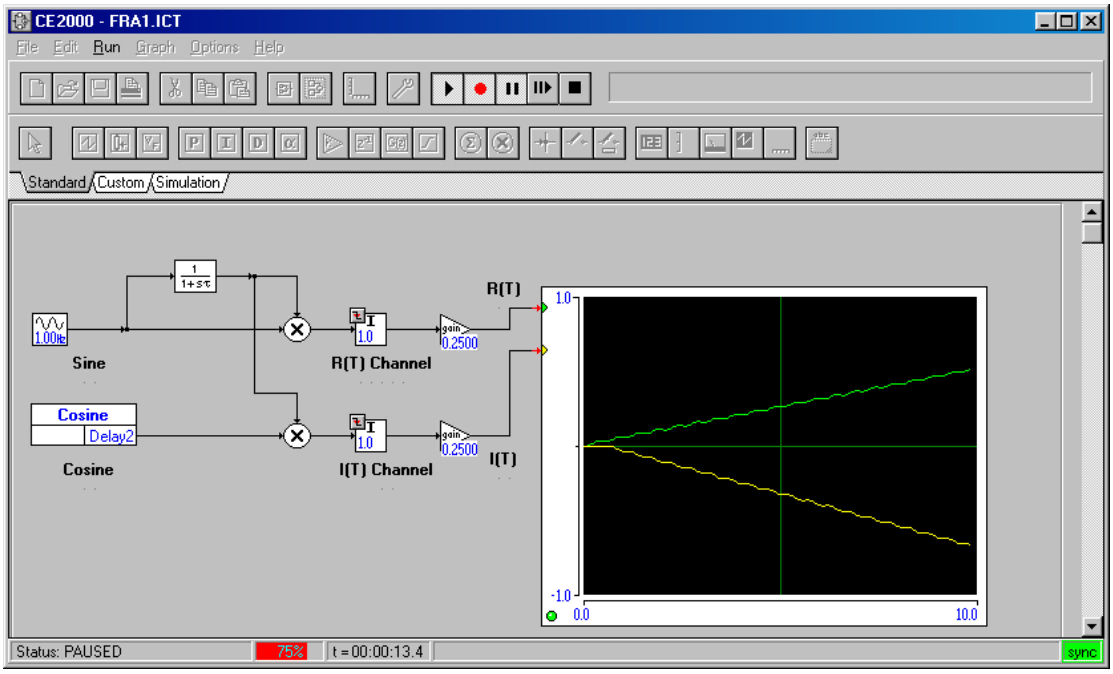

Figure 4. CE2000 implementation of a correlation FRA

Figure 4 shows a plot of $R(T)$ and $I(T)$ obtained from the CE2000 model and scaled to give the correct value over 4 cycles of the test frequency. Note that the values at 4 seconds correspond (approximately) to half the theoretical values as predicted. The values are approximate because in the basic implementation shown in Figure 4 we have not allowed the system output to reach a steady state. Thus the measurements are influenced by initial conditions. In addition the numerical integration needs to be properly designed in correlation FRA methods. It is usual to use either specialist hardware or virtual instruments to obtain accurate frequency response measurements over a wide range of frequencies.

5. A Final Word from Elke

It is a big pleasure for me to be working again with Control Systems Principles. I am come back to give some *Disziplin und Grundlichkeit* to the Control Systems Principles team. My first job was to positively influence Mark. It is difficult but I enjoy my work.

Control Systems Principles have many email requests from companies and students for help and technical support. We are sorry but Control System Principles is not able to answer questions about our white papers, unless we have a contract with your organisation. For more information about the CE2000 Control and Simulation Software go to the TQ Education and Training web site using the links on our web site www.control-systems-principles.co.uk or use the email info@tq.com. There are many books that use frequency response methods, but not many explain how to measure it. We used the notes written by our boss for Solartron Instruments. Look for this report on the 'Back Catalogue' part of the Control Systems Principles download page.

6. Reference

1. P.E. Wellstead, *Frequency Response Analysis*, Report 010/83, Solartron Instruments