BALL AND HOOP 1: Basics

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ABSTRACT: This is one of a series of white papers on systems modelling, analysis and control, prepared by Control Systems Principles.co.uk to give insights into important principles and processes in control. In control systems there are a number of generic systems and methods which are encountered in all areas of industry and technology. These white papers aim to explain these important systems and methods in straightforward terms. The white papers describe what makes a particular type of system/method important, how it works and then demonstrates how to control it. The control demonstrations are performed using models of real systems that I designed, and which have been developed for manufacture by TQ Education and Training Ltd in their CE range of equipment. This white paper is about a system that shows the dynamics of oscillating systems and non-minimum phase processes – the Ball and Hoop System.

1. What is the Ball and Hoop System?

The Ball and Hoop System is about the dynamics of a steel ball that is free to roll on the inside of a rotating circular hoop. The system is shown in Figure 1 where the ball position is based on the assumption that the hoop is rotating anti-clockwise. The hoop is mounted vertically on the shaft of an electric motor so that it can be rotated about its axis. There is a groove on the inside edge of the hoop so that a steel ball can roll freely inside the hoop. When the hoop is rotated, the ball will tend to move in the direction of hoop rotation. At some point gravity will overcome the frictional forces on the ball and will fall back. This process will repeat, causing the ball to have oscillatory motion.

The motor is used to rotate the hoop, so that its angular position can be placed under control. In the figure the angle θ is the hoop angular position. The position of the ball is given by:

1. y, the position of the ball on the hoop periphery with respect to a datum point or,
2. ψ, the slosh angle which measures the deviation of the ball from its rest position.

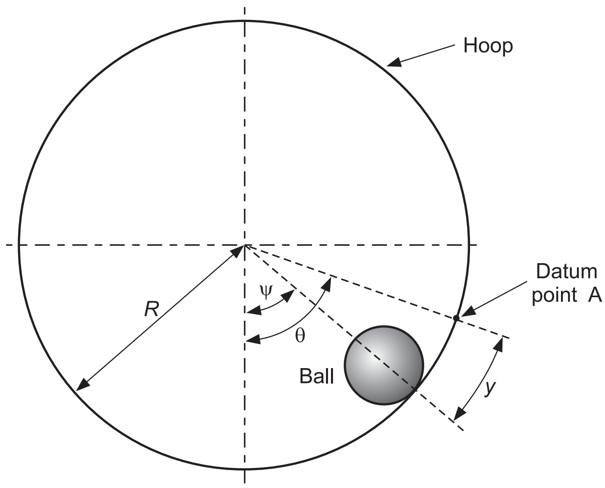

Figure 1. The Ball and Hoop System

2. Why is the Ball and Hoop System Important?

The ball and hoop system is important for two reasons: First it can be used to simulate and study the control of the oscillations of a liquid in a container when the container is moving and undergoing changes in velocity and direction. Second, it can be used to demonstrate and understand non-minimum phase behaviour in control systems. I will write about non-minimum phase systems later on in this white paper. In this section I will explain the relevance of liquid oscillations in containers for control engineering.

Oscillation in liquids is called ‘slosh’ or ‘slop’ and is important because the movement of large quantities of liquid can strongly influence the movement of the container itself – this is usually undesirable and often dangerous. For example, the movement of the liquid cargo in a road tanker as it changes direction (to go round a sharp bend for example) can alter the handling and stability of the truck. In fact any action involving the rapid movement of large quantities of fluid can exhibit the characteristic oscillations of liquid slop.

There are many other examples where liquid oscillation must be considered. Important practical examples are:

1. The liquid load in a railway wagon tanker can rock from side to side on an uneven railway track, causing undue wear to the wagon suspension and railway track.
2. The liquid cargo of a ship will slosh when the ship is in heavy seas, and this may reduce the stability of the ship.
3. The liquid fuel of a missile can oscillate when it makes a rapid change of course, and interact with the flight control systems.

In fact missile fuel is usually solid so the problem does not often exist here. In other cases, containers are designed and made to reduce or prevent liquid slosh. This does not work all the time, and liquid slosh remains an interesting control problem. In some cases, feedback control systems in a vehicle or vessel can be used to further reduce liquid slosh. Oscillating loads do not have to be in liquid form to cause difficulties, a similar problem occurs in mobile cranes, where the load at the end of a long cable can oscillate and interact with the handling of the crane and the accurate placing of the load. Here again control systems are used to reduce the oscillations of the load.

It is easy to demonstrate the dynamic behaviour of liquid slosh/slop using a deep dish or bowl that is about quarter full of water. Move the bowl quickly to one side and watch as the water moves back and forth in an oscillatory manner. If you hold the bowl while the water is oscillating you will feel the force caused by the oscillations. Imagine this force magnifed many hundreds of times as the liquid cargo of a ship moves during a storm, and you will appreciate why liquid slop is important and why it must be understood and controlled.

The ball and hoop is an analogy for the movement of liquid in a cylindrical vessel. Figure 2 shows the analogy. Figure 2(a) shows liquid slop/slosh in a cylinder, and Figure 2(b) shows the analogous rolling of a ball inside a hoop.

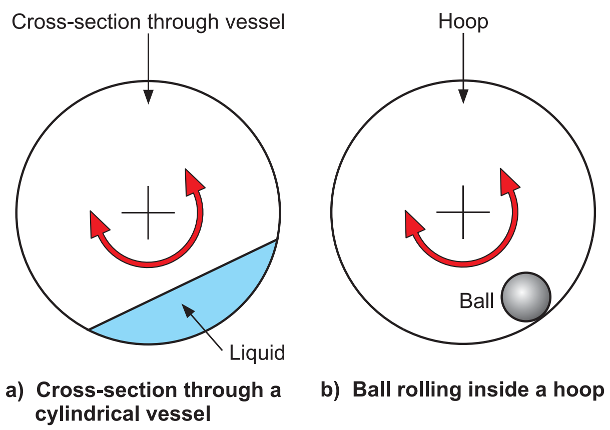

Figure 2. Illustrating the Analogy

To understand this a little better, consider the cylindrical pendulum shown in Figure 3 and compare it to the bowl of water. The radius of the cylinder, R, determines the frequency of oscillation and the rolling friction of the ball determines the decay rate of the oscillations. The same is true if the ball is replaced by a liquid – the frequency of the slop oscillations is given by the container radius, and the liquid viscosity will determine the decay rate of the oscillations.

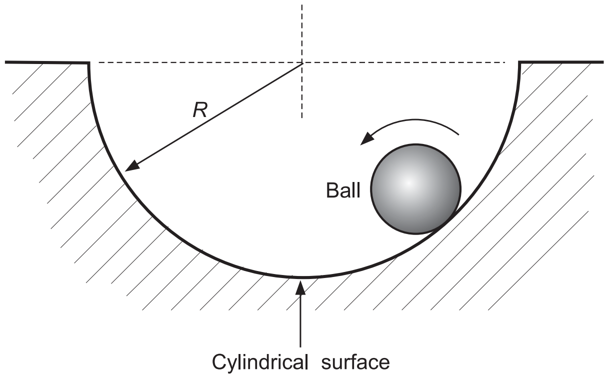

Figure 3. Cylindrical pendulum

The ball and hoop simulates an important practical problem. HOWEVER, it is also important technically for practical laboratory studies because it allows us to examine oscillations in dynamic systems in general, and to show how control can change these oscillations in interesting ways. I first thought of the ball and hoop system some years ago while working on dynamical control strategies for reducing liquid slosh. The ball and hoop was designed as a 'technology demonstrator' to illustrate to engineers and managers some control methods for reducing the oscillations of liquids in tanks – ever since it has been a surprise at how well the Ball and Hoop simulates liquid slosh, and how simple and effective anti-slosh controllers can be. The ball and hoop slosh simulator evolved into the CE109 Ball and Hoop System that is described later in this white paper.

3. The Ball and Hoop System Model

The Ball and Hoop dynamics are quite complicated to derive using the normal approach. To make it easier (for me!) I will use the variational approach. Figure 4 shows the ball and hoop again – this time with labels for all the variables that we will need. The hoop is mounted on the shaft of a motor, and the motor is assumed to be a pure source of torque $\tau(t)$. The dynamical behaviour of the system would be completely represented by the equations of motion of the angular position of the hoop, θ, and the position of y of the ball on the inner periphery of the hoop. For this reason I will use these as the generalised coordinates for Langrange's equations. Neither generalised co-ordinate is constrained so the corresponding variational co-ordinates for the system are $\delta\theta$ and δy. The Langrange equations for these coordinates are, for θ:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{\theta}}\right) - \frac{\delta L}{\delta\theta} + \frac{\delta J}{\delta\dot{\theta}} = \tau_\theta(t) \quad \textbf{(1)}$$

and for the y co-ordinate:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{y}}\right) - \frac{\delta L}{\delta y} + \frac{\delta J}{\delta\dot{y}} = F_y(t) \quad \textbf{(2)}$$

where L is the Lagrangian, J is the system co-content, $\tau_\theta(t)$ is the generalised external torque referred to the θ co-ordinate and generalised external force $F_y(t)$ referred to the y co-ordinate.

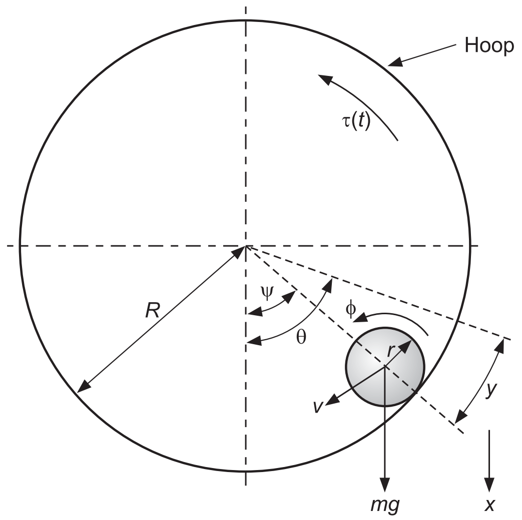

Figure 4. The Ball and Hoop System

We now need some equations that relate the generalised coordinates to other variables in the system. In particular, the angular rotation of the ball, ϕ is given by:

$$\phi = \frac{y}{r} \tag{3}$$

The translational velocity of the ball, v is given by:

$$v = (R - r)\dot{\psi} \tag{4}$$

where r = the **rolling** radius of the ball. I have highlighed rolling in red because the ball will roll in a groove in side the hoop. This means that the radius for rolling will be less than the actual radius of the ball, r_b.

The slop angle ψ is related to the generalised co-ordinates by:

$$\psi = \left(\theta - \frac{y}{R}\right) \tag{5}$$

The system Lagrangian, L, is made up entirely from the kinetic energies, U, associated with the rotation of the hoop, the rotation of the ball and the translation of the ball's centre of mass. Thus the system Lagrangian is:

$$L = U = \tfrac{1}{2}\left(I_{\mathrm{a}}(\dot{\theta})^2 + I_{\mathrm{b}}(\dot{\phi})^2 + mv^2\right) \tag{6}$$

Rewriting this equation in terms of the generalised co-ordinates gives:

$$L = \tfrac{1}{2}\left[I_{\mathrm{a}}(\dot{\theta})^2 + I_{\mathrm{b}}\left(\frac{\dot{y}}{r}\right)^2 + m\left((R - r)\left(\dot{\theta} - \frac{\dot{y}}{R}\right)\right)^2\right] \tag{7}$$

where:

I_{a} = Moment of inertia of the hoop;

I_b = Moment of inertia of the ball;

m = Mass of the ball.

In addition, the system co-content, J, is associated with the rolling friction of the ball, (friction coefficient, b_b) and the motor assembly, (rotational friction coefficient of b_m). Thus:

$$J = \frac{1}{2}\left[b_b\left(\frac{\dot{y}}{r}\right)^2 + b_m\left(\dot{\theta}\right)^2\right] \tag{8}$$

The generalised inputs are, for co-ordinate θ the generalised torque $\tau_\theta(t)$ is given by the sum of the motor input torque and gravity acting on the ball:

$$\tau_\theta(t) = \tau(t) + mg\frac{\delta x}{\delta\theta} \tag{9}$$

For the co-ordinate y, the generalised F_y is given by:

$$F_y = mg\frac{\delta x}{\delta y} \tag{10}$$

where x is the vertical displacement of the ball, measured positively in the downward direction and given by:

$$x = -(R-r)\left(1-\cos\left(\theta - \frac{y}{R}\right)\right) \tag{11}$$

Hence,

$$\tau_\theta(t) = \tau(t) - mg\,(R-r)\sin\left(\theta - \frac{y}{R}\right) \tag{12}$$

$$F_y(t) = mg\left(\frac{R-r}{R}\right)\sin\left(\theta - \frac{y}{R}\right) \tag{13}$$

The equations of motion follow from Langranges' equation, so for θ:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{\theta}}\right) - \frac{\delta L}{\delta\theta} + \frac{\delta J}{\delta\dot{\theta}} = \tau_\theta(t) \tag{14}$$

$$\left[I_a + m(R-r)^2\right]\ddot{\theta} + b_m\dot{\theta} - \frac{m(R-r)^2}{R}\ddot{y} = \tau(t) - mg\,(R-r)\sin\left(\theta - \frac{y}{R}\right) \tag{15}$$

For the y co-ordinate:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{y}}\right) - \frac{\delta L}{\delta y} + \frac{\delta J}{\delta\dot{y}} = F_y(t) \tag{16}$$

$$\left[\frac{I_b}{r^2} + \frac{m(R-r)^2}{R^2}\right]\ddot{y} + \frac{b_b}{r^2}\dot{y} - \frac{m(R-r)^2}{R}\ddot{\theta} = mg\frac{(R-r)}{R}\sin\left(\theta - \frac{y}{R}\right) \tag{17}$$

Equations (15) and (17) are the dynamical equations of motion of the Ball and Hoop and can be combined in one matrix differential equation set to give:

$$\begin{pmatrix} I_a + m(R-r)^2 & \frac{-m(R-r)^2}{R} \\ \frac{-m(R-r)^2}{R} & \frac{I_b}{r^2} + \frac{m(R-r)^2}{R^2} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} b_m & 0 \\ 0 & \frac{b_b}{r^2} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{y} \end{pmatrix} + mg \begin{pmatrix} (R-r)\sin(\theta - \frac{y}{R}) \\ -\left(\frac{R-r}{R}\right)\sin(\theta - \frac{y}{R}) \end{pmatrix} = \begin{pmatrix} \tau(t) \\ 0 \end{pmatrix} \quad \textbf{(18)}$$

4. Model Simplifications

4.1. Linearization

By assuming that the slop angle $\psi = \left(\theta - \frac{y}{R}\right)$ is relatively small, we can replace $\sin\left(\theta - \frac{y}{R}\right)$ by $\left(\theta - \frac{y}{R}\right)$ to get a linear second order matrix equation of the form:

$$\begin{pmatrix} I_a + m(R-r)^2 & \frac{-m(R-r)^2}{R} \\ \frac{-m(R-r)^2}{R} & \frac{I_b}{r^2} + \frac{m(R-r)^2}{R^2} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} b_m & 0 \\ 0 & \frac{b_b}{r^2} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{y} \end{pmatrix} + mg \begin{pmatrix} (R-r) & -\frac{(R-r)}{R} \\ -\frac{(R-r)}{R} & \frac{(R-r)}{R^2} \end{pmatrix} \begin{pmatrix} \theta \\ y \end{pmatrix} = \begin{pmatrix} \tau(t) \\ 0 \end{pmatrix} \quad \textbf{(19)}$$

The equation (19) is a specific form of the equation:

$$M\ddot{X} + D\dot{X} + KX = Bu \quad \textbf{(20)}$$

This general form is often used in robotics to neatly formulate the differential equations of a system. It is also a convenient way to formulate state space equations for the system. Specifically by putting $X = (\theta \quad y)^T$ and $X_1 = X, X_2 = \dot{X}$, then the state equations from equation (20) are:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix} u$$
$$Y = C \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \textbf{(21)}$$

The choice of the 'read out' matrix C will depend upon the measured states. The equation set (21) can be used in setting up state space controller designs –this will be done in a possible future white paper.

4.2. Approximations and Substitutions

Because the hoop radius is much bigger than the ball radius, e.g. $R >> r$, then equation (19) can be written:

$$\begin{pmatrix} I_a + mR^2 & -mR \\ -mR & \dfrac{I_b}{r^2} + m \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} b_m & 0 \\ 0 & \dfrac{b_b}{r^2} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{y} \end{pmatrix} + mg\begin{pmatrix} R & -1 \\ -1 & \dfrac{1}{R} \end{pmatrix}\begin{pmatrix} \theta \\ y \end{pmatrix} = \begin{pmatrix} \tau(t) \\ 0 \end{pmatrix} \qquad \textbf{(22)}$$

Also the moment of inertia of a solid ball is $I_b = \dfrac{2}{5} m r_b^2$, (note the ball radius and rolling radius are different) so that the equations become:

$$\begin{pmatrix} I_a + mR^2 & -mR \\ -mR & m\left(\dfrac{2r_b^2}{5r^2} + 1\right) \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} b_m & 0 \\ 0 & \dfrac{b_b}{r^2} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{y} \end{pmatrix} + mg\begin{pmatrix} R & -1 \\ -1 & \dfrac{1}{R} \end{pmatrix}\begin{pmatrix} \theta \\ y \end{pmatrix} = \begin{pmatrix} \tau(t) \\ 0 \end{pmatrix} \qquad \textbf{(23)}$$

The matrix differential equations can sometimes be separated. In particular, if the inertial torque of the ball, m, is small compared to that of the hoop and the motor torque, then the first equation in (23) becomes the well known differential equation for a DC motor with an inertial load, I_a, and viscous friction b_m:

$$I_a \ddot{\theta} + b_m \dot{\theta} = \tau(t) \qquad \textbf{(24)}$$

and the second equation in (23) is:

$$\left(\frac{2r_b^2}{5r^2} + 1\right)\ddot{y} + \frac{b_b}{mr^2}\dot{y} + \frac{g}{R} y = R\left(\ddot{\theta} + \frac{g}{R}\theta\right) \qquad \textbf{(25)}$$

These two separate differential equations are useful because they let us write the system model as two cascaded transfer functions, where in the first transfer function the motor torque produces a hoop angle, and in the second transfer function the hoop angle produces a ball position. We will look at this and other forms of the system models in Section 5.

5. Nice Features of the Ball and Hoop System Dynamics

Two special features of the ball and hoop are the ability to demontrate ‘zeros of transmission’ and to show ‘non-minimum phase behaviour’.

5.1. Zeroes of Transmission

If the equation (25) is written as a transfer function

$$\frac{y(s)}{\theta(s)} = R\left[\frac{s^2 + \dfrac{g}{R}}{\left(\dfrac{2r_b^2}{5r^2} + 1\right)s^2 + \dfrac{b_b}{mr^2}s + \dfrac{g}{R}}\right]$$

Then it is clear that the transfer function has purely imaginary zeroes at $s = \pm j\dfrac{g}{R}$. This means that if the hoop angle θ is under feedback control and a sine wave of frequency $\dfrac{g}{R} radians/\sec$ is applied to θ, then there will be exact zero response from the ball position output $y(s)$.In physical terms this corresponds to the case where the ball oscillates inside the hoop at exactly the same frequency as the hoop. Thus to an observer standing on the hoop, the ball does not move because the ball and hoop are moving in exact synchronism. Of course the ball is in fact really moving when looked from the outside – it is just that from a particular measurement/observation point it appears to be stationary.

5.2. Non-Minimum Phase Behavior and Shifting Zeroes

The outputs for the Ball and Hoop are $\theta(s)$ and $\psi(s)$. From these it is possible to construct the signal $y(s)$ by subtracting scaled measurements of $\theta(s)$ and $\psi(s)$. Scaling is required to take account of the amplification and possible sign inversion introduced by the angle sensors. Generally, we can consider the synthetic output signal $x(s)$ is given by:

$$x(s) = \theta(s) - k_s \psi(s) \tag{26}$$

where:

k_s = Scalar gain factor.

Note that when k_s is unity, $x(s)$ is the variable $\frac{y(s)}{R}$, and this corresponds to a scaled version of the ball position on the periphery of the hoop. Note that Equation (26) is the combination of two system output signals, such that the input has two paths to the output $x(s)$. Rewriting Equation (26) as a transfer function gives:

$$\frac{x(s)}{\theta(s)} = 1 - k_s \frac{\psi(s)}{\theta(s)}$$

By varying the gain k_s a root locus of the transfer function zeroes can be plotted and part of the root locus is in the right hand plane. This means that non-minimum phase behaviour occurs in the system. This will give the characteristic feature of non-minimum phase where the output goes initially in the ‘wrong’ direction when step inputs are applied.

5.3. Slop Control

When a step change in desired hoop angle is made the ball can oscillate before settling to its new rest position – e.g. it exhibits slop. By feeding back a component of the slop angle $\psi(s)$, it is possible to suppress the ball oscillations and thus illustrate a dynamic form of slop control. I will not demonstrate this here but hopefully we can do it in a future white paper.

4. Example of a Ball and Hoop System

The CE109 Ball and Hoop from TQ Education and Training Ltd (Figure 4) is a desk top version of the ball and hoop problem for teaching and research in oscillatory systems, feedback control, non-minimum phase systems and other dynamic systems properties. The main hardware elements are:

1. The hoop which can rotate with a steel ball on its inner periphery. (The black disc in the centre of the figure.
2. The servomotor, M, which drives the hoop and controls the hoop angle.
3. A hoop angle sensor. (The white arrow painted on the black disc is a visual indication of hoop angle.
4. An angle sensor for the ball in the hoop. (The black pointer pointing vertically down is a visual indication of ball angle).

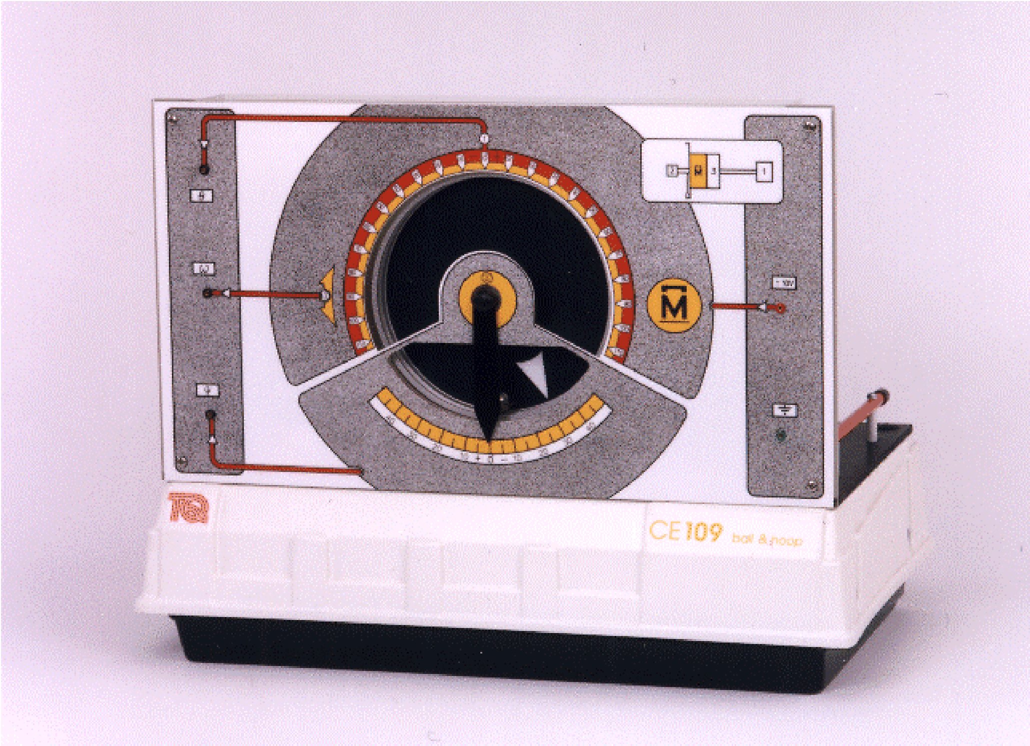

Figure 4. The CE109 Ball and Hoop System

The CE109 Ball and Hoop System can be used to demonstrate velocity and position control of the hoop. When the hoop angle is under control the way is open to study the oscillatory dynamics of slop (as explained in the previous example) and to examine ways of controlling the amount of slop.

6. A Final Word

Elke and colleagues (but especially Elke) put a final word at the end of the white papers – so I will do the same. I hope that you have got some ideas about ball and hoop systems. I have not included any control systems examples in this white paper – mainly because we wanted to explain the dynamics and we did not have a CE109 available at the time of writing. However, we hope that a visiting researcher will appear to write a white paper especially on ball and hoop control – so check the web site from time to time. I am sorry to say that it is not possible to answer general questions from students and engineers about the contents of our white papers, unless we have an arrangement with your organisation. For more information about the ball and hoop go to the TQ Education and Training Ltd web site using the links on our web site www.control-systems-principles.co.uk or use the email info@tq.com.